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The Riemann Problem for a van der Waals Fluid with Entropy Rate Admissibility Criterion—Nonisothermal Case

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1. INTRODUCTION

In this paper we extend the applicability of the entropy rate admissibility criterion proposed by Dafermos [1, 2] to the Riemann problem of a van der Waals fluid for the nonisothermal case. The applicability of this criterion to the isothermal case of the above fluid has been discussed in an earlier paper [3].

The above criterion was originally proposed for hyperbolic systems of conservation laws of the form

$$w_t + f(w)_x = 0. \tag{1.1}$$

It is well known that in general the weak solutions (bounded measurable functions which satisfy (1.1) in the sense of distributions) are not unique. In order to select a physically relevant solution, various admissibility criteria have been proposed. The entropy rate admissibility criterion, one of these criteria, roughly says that for the admissible solution, the entropy decreases with the highest rate. In other words, a solution $w(x, t)$ will be called admissible if there is no solution $\bar{w}(x, t)$ with the property that for some $\tau \in [0, T]$, $w(x, t) = \bar{w}(x, t)$ on $(-\infty, \infty) \times [0, \tau]$ and $D_+ H_{\bar{w}}(\tau) < D_+ H_w(\tau)$, where $\eta(w)$ is a strictly convex entropy and

$$H_w(\tau) = \int_{-\infty}^{\infty} \eta(w(x, t))\, dt. \tag{1.2}$$

A mathematical interest of the Riemann problem for a van der Waals fluid is that the system we will treat is of hyperbolic-elliptic mixed type. For this type of nonhyperbolic system, a physically motivated criterion has

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been proposed by Slemrod [4, 5]. His argument is based on the capillarity effect of fluids, which is an extension of the work of Serrin [6].

We shall show that the entropy rate admissibility criterion is applicable to the Riemann problem of the above nonhyperbolic system. Specifically, we apply this criterion to a small class of solutions consisting of a backward wave, a forward wave, a contact discontinuity, and a phase boundary, but not to all possible solutions. We should notice that this criterion is applied in a limited sense.

This paper consists of five sections. In Section 2, we discuss the nature of a van der Waals fluid and the main assumptions of the system we treat. In Section 3, we formulate the Riemann problem and then describe briefly the elementary waves which arise in the problem, namely, rarefaction waves, shocks, contact discontinuities and phase boundaries. We show, in Section 4, that there exists a one-parameter family of solutions to the Riemann problem in consideration. Then, we show that the consequence of the entropy rate admissibility criterion agrees with the result of classical thermodynamics. Namely, we compute the first and second derivatives of the entropy rate with respect to the parameter to see that the stationary phase boundary is admissible if the physical entropies on the left and on the right of the phase boundary at $t=0$ are equal and is not admissible if they are not equal. In Section 5, in order to enforce the applicability of the entropy rate admissibility criterion, we show an example of nontrivial solution which minimizes the entropy rate locally among the solutions assumed in Section 3.

2. van der Waals Fluid

We consider the system of a one-dimensional fluid flow given by

$$\begin{aligned} u_t + p_x &= 0, \\ v_t - u_x &= 0, \\ E_t + (pu)_x &= 0, \end{aligned} \tag{2.1}$$

where u, v, E, and p are the velocity, the specific volume, the total energy, and the pressure of the fluid. Here the total energy is given by $E=\frac{1}{2}u^2+e$, where e is the internal energy. This system expresses the nonisothermal flow of an inviscid and compressible fluid in Lagrangian coordinates. Unlike the ideal gas in which the pressure is given by $p=R\theta/v$, we assume that the constitutive relation is similar to that of a van der Waals fluid, namely,

$$p=\frac{R\theta}{v-b}-\frac{a}{v^2}, \tag{2.2}$$

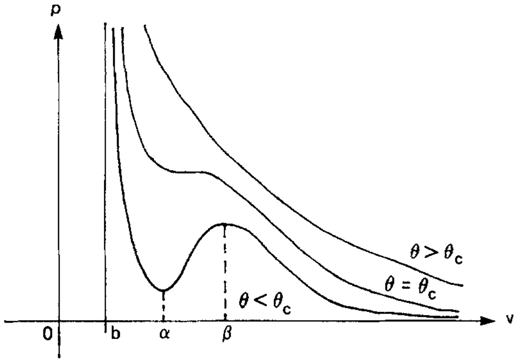

FIG. 1. Isotherms for different values of θ.

where R, a, and b are positive constants and θ is the temperature. In Fig. 1 we sketch a few isotherms of (2.2) for different values of θ. If the temperature is below the critical temperature $\theta_c = 8a/27bR$, the isotherm has the following features:

$$\begin{aligned} &\text{(i)} \quad p_v(v, \theta_0) < 0 \text{ on } (b, \alpha) \cup (\beta, \infty), \\ &\text{(ii)} \quad p_v(v, \theta_0) > 0 \text{ on } (\alpha, \beta), \\ &\text{(iii)} \quad p_v(\alpha, \theta_0) = p_v(\beta, \theta_0) = 0. \end{aligned} \tag{2.3}$$

The domain (b, α) is called the α-phase (the liquid phase) and the domain (β, ∞) is called the β-phase (the vapor phase). The domain (α, β) is assumed to be unstable and is referred to as the spinodal region.

Although we have employed v and θ as the state variables for the pressure, in thermodynamics it is assumed that the pressure can be expressed also as a function of (v, e) or (v, s), namely,

$$p = p(v, \theta) = p(v, e) = p(v, s). \tag{2.4}$$

In what follows, we mainly use (v, e) or (v, s) as the state variables for the pressure. In these cases we assume that

$$p_e(v, e) > 0, \qquad p_s(v, s) > 0 \tag{2.5}$$

hold even in the nonhyperbolic region. Another important relation is

$$p_v(v, s) = -pp_e(v, e) + p_v(v, e), \tag{2.6}$$

which can be obtained from the thermodynamic relation

$$de = -p\, dv + \theta\, ds. \tag{2.7}$$

Hereafter, we denote $p_v(v, s)$, $p_v(v, e)$, $p_s(v, s)$, and $p_e(v, e)$ by p_v, $\bar{p}_v$, p_s, and p_e, respectively.

Use of (2.6) shows that the characteristic speeds for the system (2.1) can be expressed as

$$\lambda_1 = -\sqrt{-p_v}, \qquad \lambda_2 = 0, \qquad \lambda_3 = \sqrt{-p_v}. \tag{2.8}$$

Or if we use (v, θ) as the state variables,

$$\lambda_1^2, \lambda_3^2 = -p_v(v, \theta) + \frac{\theta p_\theta^2}{e_\theta}, \qquad \lambda_2 = 0. \tag{2.9}$$

We assume that there is a region of v and s in which $p_v(v, s)$ is positive, so that the system becomes nonhyperbolic. As the relation $e_\theta > 0$ is a physically relevant assumption, we see, from (2.8) and (2.9), that the existence of the region where $p_v > 0$ guarantees the existence of the spinodal region. This conversely implies that outside the spinodal region p_v must be negative.

3. The Riemann Problem and Elementary Waves

We consider the Riemann problem for system (2.1). The Riemann problem is a special initial value problem in which the initial condition is given by

$$w^T = (u, v, e)(x, 0) = \begin{cases} (u_0, v_0, e_0), & x < 0, \\ (u_1, v_1, e_1), & x > 0, \end{cases} \tag{3.1}$$

where the right-hand sides are constants. In this paper we furthermore require that

$$\begin{aligned} &\text{(i)} \quad v_0 \text{ is in the } \alpha\text{-phase and } v_1 \text{ is in the } \beta\text{-phase,} \\ &\text{(ii)} \quad p_0 = p_1, u_0 = u_1. \end{aligned} \tag{3.2}$$

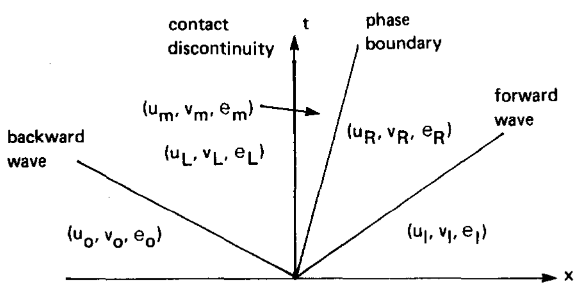

FIG. 2. A possible solution configuration.

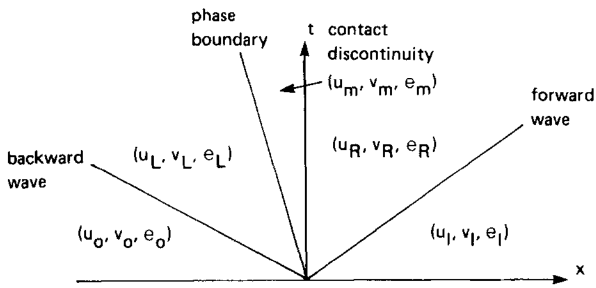

FIG. 3. A possible solution configuration.

Therefore, the phase change takes place at $x=0$ in the initial data. In this case we assume that the solution consists of four different types of elementary waves, namely, a rarefaction wave, a shock, a contact discontinuity, and a phase boundary and is given as Fig. 2 or Fig. 3. In these figures a wave means either a rarefaction wave or a shock and a forward (backward) wave means a wave with a positive (negative) speed. We should notice that the phase boundary and the contact discontinuity may coalesce (Fig. 4).

Remark 3.1. As it is easier to use (u, v, e) in order to obtain the differential equation (3.9) and (3.10) from the Rankine–Hugoniot condition, and it is necessary to use the internal energy in our discussion, we employ (u, v, e) as the state variables.

It is now in order to explain the four types of waves briefly:

(a) Rarefaction wave: This is a continuous solution $w(\xi)$, $\xi = x/t$, of (2.1). The set of $w=(u, v, e)^T$ forms a one-parameter family of states which can be connected to (u_0, v_0, e_0) on the right by a rarefaction wave. This set satisfies the differential equation

$$\frac{dw}{d\xi} = r_k \qquad (k = \mathrm{B}, \mathrm{F}), \tag{3.3}$$

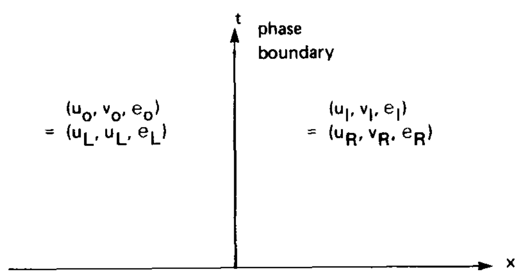

FIG. 4. A possible solution configuration.

where r_k ($k = \mathrm{B}, \mathrm{F}$) are the right eigenvectors of the Jacobian of $f(w)$ which are given by

$$\begin{aligned} r_{\mathrm{B}} &= \left(1, \frac{1}{\sqrt{-p_v}}, -\frac{p}{\sqrt{-p_v}}\right)^T, \\ r_{\mathrm{F}} &= \left(1, -\frac{1}{\sqrt{-p_v}}, \frac{p}{\sqrt{-p_v}}\right)^T. \end{aligned} \tag{3.4}$$

Here, we use the normalization so that $\xi = u - u_0$. The integral curves $R(w_0)$ of (3.2) are called rarefaction curves. The subscript $k = \mathrm{B}$ ($k = \mathrm{F}$) corresponds to a backward (forward) rarefaction curve. It is obvious that rarefaction curves are defined in the hyperbolic region.

A smooth function of (u, v, e) is called a k-Riemann invariant if it is constant along an R_k curve. It is well known that for n-conservation laws there exist, at least locally, $n-1$ linearly independent k-Riemann invariants for each k. For system (2.1) they are given by

$$\begin{aligned} k = \mathrm{B}: &\qquad s \text{ and } u - \int^v \sqrt{-p_v}, \\ k = \mathrm{F}: &\qquad s \text{ and } u + \int^v \sqrt{-p_v}. \end{aligned} \tag{3.5}$$

(b) Shock: A shock is a jump discontinuity across which the Rankine–Hugoniot condition is satisfied. The condition is given by

$$\begin{aligned} \sigma(u - u_0) &= p - p_0, \\ \sigma(v - v_0) &= -(u - u_0), \\ \sigma(E - E_0) &= pu - p_0 u_0, \end{aligned} \tag{3.6}$$

where σ is the speed of propagation of the jump discontinuity and (u, v, e) and (u_0, v_0, e_0) are the states on the right and on the left of jump discontinuity, respectively. Combining (3.6a) and (3.6b), we find that

$$(v - v_0)(p - p_0) = -(u - u_0)^2. \tag{3.7}$$

The third equation in (3.6) can be rewritten equivalently as

$$e - e_0 = -\tfrac{1}{2}(p + p_0)(v - v_0). \tag{3.8}$$

As in the rarefaction waves, the set of $w = (u, v, e)^T$ forms a one-parameter family of states which can be connected to (u_0, v_0, e_0) on the right by a shock. This set is called a shock curve. Following Liu [7], if we differentiate (3.6) with respect to the parameter $\xi(=u-u_0)$, we obtain the differential equation

$$\frac{dw}{d\xi}=h_k \qquad (k=\mathrm{B},\mathrm{F}), \tag{3.9}$$

where

$$h_k=\left(1,\frac{2\sigma_k-p_e(u-u_0)}{p_v-\sigma_k^2+p_e(p-p_0)},\frac{(u-u_0)(p_v+\sigma_k^2)+(u-u_0)\,pp_e-2\sigma_k p}{p_v-\sigma_k^2+p_e(p-p_0)}\right)^T, \tag{3.10}$$

and

$$\sigma_\mathrm{F}=-\sigma_\mathrm{B}=\sqrt{-\frac{p-p_0}{v-v_0}}. \tag{3.11}$$

It is well known that the entropy increases across a shock and that the change of entropy is of the third order in the parameter change, namely, $s-s_0=O(\xi^3)$.

(c) Contact discontinuity: This is another type of jump discontinuity across which the relation

$$p=p_0, \qquad u=u_0, \qquad \sigma=0 \tag{3.12}$$

is satisfied. It is easy to see that if we set $\sigma=0$ in the Rankine–Hugoniot condition, we can obtain the above relation.

(d) Phase boundary: This is, also, a jump discontinuity across which the Rankine–Hugoniot condition is satisfied. The main difference between a shock and a phase boundary is that the phase change takes place across a phase boundary, but on the other hand, the phase is the same across a shock. We can define a phase boundary curve in the analogous manner as the shock curve. Nevertheless, since v and v_0 are in the different phases, the phase boundary curve for a given (u_0, v_0, e_0) will not pass through this point.

4. Comparison of the Entropy Rate

For system (2.1) the entropy rate $D_+H(\tau)$ is given by

$$D_+H(\tau)=\sum_{\substack{\text{jump}\\ \text{discontinuities}}}\sigma(\tau)(s_+-s_-), \tag{4.1}$$

where $\sigma(\tau)$ is the speed of propagation of each jump discontinuity, and s_+ and s_- are the entropy on the right and on the left of a jump discontinuity, respectively; see Hsiao [8] and Hattori [9]. As D_+H does not depend on τ in the Riemann problem, we use Φ for the entropy rate of the system (2.1) and use the subscript to denote the entropy rate for each jump discontinuity (for example, Φ_B means the entropy rate for the backward shock). In this section we show that the solution configurations in Figs. 2, 3, and 4 form a one-parameter family of olutions. Namely, the solution types in Figs. 2 or 3 will emerge from the solution coniguration in Fig. 4 as we change the parameter. We then show that the derivatives of the entropy rate with respect to the parameter for the solution configuration in Fig. 4 have different slopes depending on the initial data. We shall distinguish two cases:

(i) $s_0 > s_1$ or $s_0 < s_1$,

(ii) $s_0 = s_1$,

where $s = s_0$ for $x < 0$ and $s = s_1$ for $x > 0$ are the entropy at $t = 0$. In either case the set of (u_L, v_L, e_L) forms a one-parameter family of states which can be connected to (u_0, v_0, e_0). We denote the parameter by ξ_L $(= u_L - u_0)$. In the same manner, the set of (u_R, v_R, e_R) forms a one-parameter family of states which can be connected to (u_1, v_1, e_1). We denote the parameter by ξ_R $(= u_R - u_1)$. Hence,

$$\begin{aligned}(u_L, v_L, e_L) &= (u_L(\xi_L), v_L(\xi_L), e_L(\xi_L)),\\ (u_R, v_R, e_R) &= (u_R(\xi_R), v_R(\xi_R), e_R(\xi_R)).\end{aligned} \tag{4.2}$$

Remark 4.1. As we will see later in this section, ξ_R will be a function of ξ_L. Also, if we increase ξ_L from zero, then the solution configuration of Fig. 2 type, namely, the solution which has the forward phase boundary, will emerge from the solution configuration of Fig. 4. On the other hand, if we decrease ξ_L from zero, then the solution configuration of Fig. 3 type, namely, the solution which has the backward phase boundary, will emerge from the solution configuration of Fig. 4.

Remark 4.2. Whether or not physically we have cases (i) and (ii) is an interesting question. If we use the van der Waals state equation given by (2.2) and assume that the entropy and the internal energy are given, respectively, by

$$\begin{aligned}s &= R\ln(v-b) + C_v \ln\theta + C_v,\\ e &= C_v\theta - a/v + \text{constant},\end{aligned}$$

then the pressure is given by

$$p(v, s) = \frac{R\exp(s/C_v - 1)}{(v-b)^{1+R/C_v}} - \frac{a}{v^2}$$

(see [5, 10] for the detailed discussion). Since C_v is greater than R, it is easy to see that if the entropy is a small constant s_c, then $p(v, s_c)$ has essentially the same features as (2.3). Therefore, in this case we can satisfy all the requirements for the initial data, for example, $u_0 = u_1$, $p_0 = p_1$, and $s_0 = s_1$, etc., by choosing appropriate initial data. It should also be remarked that we treat the case where the entropy is close to s_c so that v_1 is greater than v_0.

Now, we begin the discussion with the case (i).

(i) $s_0 > s_1$ or $s_0 < s_1$.

First, we treat the case where we have the forward phase boundary (Fig. 2). In this case, as the state (u_m, v_m, e_m) is connected to (u_R, v_R, e_R) by the forward phase boundary, from the Rankine–Hugoniot condition we have the relation

$$\begin{aligned} (v_R(\xi_R) - v_m)(p_R(\xi_R) - p_L(\xi_L)) + (u_R(\xi_R) - u_L(\xi_L))^2 &= 0, \\ e_R(\xi_R) - e_m + \tfrac{1}{2}(p_R(\xi_R) + p_L(\xi_L))(v_R(\xi_R) - v_m) &= 0, \\ p_m(v_m, e_m) - p_L(\xi_L) &= 0. \end{aligned} \tag{4.3}$$

By the implicit function theorem, it is easy to show that ξ_R, v_m, and e_m are expressed as functions of ξ_L near $\xi_L = 0$. In what follows, we use $\cdot$ for the derivatives with respect to the parameter ξ_L and use $'$ for the derivatives with respect to each parameter. For example,

$$\dot{p}_R = \frac{dp_R}{d\xi_L} = \frac{dp_R}{d\xi_R}\frac{d\xi_R}{d\xi_L} = p'_R \dot{\xi}_R, \qquad \dot{p}_L = p'_L,$$

etc. Using the original Rankine–Hugoniot condition, we have

$$\begin{aligned} \sigma_P(u_R - u_m) &= p_R - p_m, \\ \sigma_P(v_R - v_m) &= -(u_R - u_m), \\ e_R - e_m &= -\tfrac{1}{2}(p_R + p_m)(v_R - v_m), \\ p_m = p_L, &\qquad u_m = u_L. \end{aligned} \tag{4.4}$$

Differentiating (4.4) with respect to ξ_L regarding ξ_R, v_m, and e_m as functions of ξ_L, we obtain

$$
\begin{aligned}
\dot{\sigma}_{\mathrm{P}}(u_{\mathrm{R}}-u_{\mathrm{L}})+\sigma_{\mathrm{P}}(\dot{\xi}_{\mathrm{R}}-1)&=p'_{\mathrm{R}}\dot{\xi}_{\mathrm{R}}-p'_{\mathrm{L}},\\
\dot{\sigma}_{\mathrm{P}}(v_{\mathrm{R}}-v_m)&=1-\dot{\xi}_{\mathrm{R}}-\sigma_{\mathrm{P}}(v'_{\mathrm{R}}\dot{\xi}_{\mathrm{R}}-\dot{v}_m),\\
e'_{\mathrm{R}}\dot{\xi}_{\mathrm{R}}-\dot{e}_m&=-\tfrac{1}{2}(p'_{\mathrm{R}}\dot{\xi}_{\mathrm{R}}+p'_{\mathrm{L}})(v_{\mathrm{R}}-v_m)\\
&\quad-\tfrac{1}{2}(p_{\mathrm{R}}+p_{\mathrm{L}})(v'_{\mathrm{R}}\dot{\xi}_{\mathrm{R}}-\dot{v}_m),\\
\dot{p}_m&=\bar{p}^m_v\dot{v}_m+p^m_e\dot{e}_m=p'_{\mathrm{L}},
\end{aligned}
\tag{4.5}
$$

where p^m_e is the derivative of p_m with respect to e, etc.

Eliminating $\dot{\sigma}_{\mathrm{P}}$ from (4.5a) and (4.5b) and, also, eliminating $\dot{e}_m$ from (4.5c) and (4.5d), we find that

$$
\begin{pmatrix}
\sigma^2_{\mathrm{P}} & (-2\sigma_{\mathrm{P}}+p'_{\mathrm{R}}-\sigma^2_{\mathrm{P}}v'_{\mathrm{R}})\\
\{\bar{p}^m_v-\frac{1}{2}p^m_e(p_{\mathrm{R}}+p_{\mathrm{L}})\} & \{p^m_e e'_{\mathrm{R}}+\frac{1}{2}p^m_e(v_{\mathrm{R}}-v_m)\,p'_{\mathrm{R}}+\frac{1}{2}\,p^m_e(p_{\mathrm{R}}+p_{\mathrm{L}})v'_{\mathrm{R}}\}
\end{pmatrix}
\begin{pmatrix}\dot{v}_m\\ \dot{\xi}_{\mathrm{R}}\end{pmatrix}
=\begin{pmatrix}
p'_{\mathrm{L}}-2\sigma_{\mathrm{P}}\\
p'_{\mathrm{L}}\{1-\frac{1}{2}p^m_e(v_{\mathrm{R}}-v_m)\}
\end{pmatrix}.
\tag{4.6}
$$

Denote as D_1 the determinant of the matrix on the left-hand side of (4.6). It is easy to see that D_1 is not zero at $\xi_{\mathrm{L}}=0$ ($\sigma_{\mathrm{P}}=0$).

Now, we compute the derivatives of the entropy rate with respect to ξ_{L}. The entropy rate of a backward shock, a forward shock, and a phase boundary are given, respectively, by

$$
\begin{aligned}
\Phi_{\mathrm{B}}&=\sigma_{\mathrm{B}}(s_{\mathrm{L}}-s_0),\\
\Phi_{\mathrm{F}}&=\sigma_{\mathrm{F}}(s_1-s_{\mathrm{R}}),\\
\Phi_{\mathrm{P}}&=\sigma_{\mathrm{P}}(s_{\mathrm{R}}-s_m).
\end{aligned}
\tag{4.7}
$$

Therefore, for example, if there are a backward rarefaction wave and a forward shock in the solution, the entropy rate is $\Phi=\Phi_{\mathrm{P}}+\Phi_{\mathrm{F}}$. As $s_{\mathrm{L}}-s_0=O(\xi^3_{\mathrm{L}})$ and $s_1-s_{\mathrm{R}}=O(\xi^3_{\mathrm{R}})=O(\xi^3_{\mathrm{L}})$, the slope of Φ_{P} will determine the slope of the entropy rate Φ at $\xi_{\mathrm{L}}=0$. The differentiation of Φ_{P} with respect to ξ_{L} and use of (4.5b) imply

$$
\dot{\Phi}_{\mathrm{P}}=\{(1-\dot{\xi}_{\mathrm{R}})-\sigma_{\mathrm{P}}(\dot{v}_{\mathrm{R}}-\dot{v}_m)\}\left(\frac{s_{\mathrm{R}}-s_m}{v_{\mathrm{R}}-v_m}\right)+\sigma_{\mathrm{P}}(\dot{s}_{\mathrm{R}}-\dot{s}_m).
\tag{4.8}
$$

From (4.6) we see

$$
\dot{\xi}_{\mathrm{R}}=\frac{p'_{\mathrm{L}}A+O(\sigma_{\mathrm{P}})}{p'_{\mathrm{R}}A+O(\sigma_{\mathrm{P}})},
\tag{4.9}
$$

where $A = \bar{p}_v^m - \frac{1}{2} p_e^m (p_R + p_L)$. As $\sigma_P = 0$ at $\xi_L = 0$ (because $p_0 = p_L = p_m = p_r = p_1$ at $\xi_L = 0$),

$$\dot{\xi}_R \big|_{\xi_L = 0} = \frac{p'_L}{p'_R}\bigg|_{\xi_L = 0, \xi_R = 0} = -\frac{\sqrt{-p_v^0}}{\sqrt{-p_v^1}} < 0 \tag{4.10}$$

for all the cases (there are two types of backward waves and two types of forward waves). From (4.5) and (4.6)

$$\sigma_P = O(\xi_L), \qquad \dot{v}_m = O(1).$$

Also, from the relation

$$\dot{p}_m = \dot{p}_L = p'_L = p_v^m \dot{v}_m + p_s^m \dot{s}_m, \tag{4.11}$$

we can easily see that

$$\dot{s}_m = O(1).$$

Hence, if $s_R \neq s_m$, the term $(1 - \dot{\xi}_R)$ determines the sign of $\dot{\Phi}$ at $\xi_L = 0$, namely,

$$\begin{aligned} &\dot{\Phi}_P > 0 \qquad \text{at} \quad \xi_L = 0 \text{ if } s_1 = s_R > s_m = s_L = s_0 \text{ at } \xi_L = 0, \\ &\dot{\Phi}_P < 0 \qquad \text{at} \quad \xi_L = 0 \text{ if } s_R < s_m \text{ at } \xi_L = 0. \end{aligned} \tag{4.12}$$

In order to know whether ξ_L is positive or negative, we use (4.4b). As we have the forward phase boundary, the left-hand side of (4.4b) is positive. This means that $u_R \leqslant u_m$ should hold. Using the relation

$$u_1 = u_R = u_m = u_L = u_0 \qquad \text{at} \quad \xi_L = 0, \quad u_m = u_L,$$
$$\xi_R = u_R - u_1, \quad \xi_L = u_L - u_0,$$

we see that

$$u_m - u_R = (u_L - u_0) - (u_R - u_1) = \xi_L - \xi_R \geqslant 0.$$

As ξ_R has the opposite sign to ξ_L from (4.10), we conclude that ξ_L should increase for the forward phase boundary.

Next, we discuss the case where we have the backward phase boundary (Fig. 3). The Rankine–Hugoniot condition reads

$$\begin{aligned} \sigma_P(u_m - u_L) &= p_m - p_L, \\ \sigma_P(v_m - v_L) &= -(u_m - u_L), \\ e_m - e_L &= -\tfrac{1}{2}(p_m + p_L)(v_m - v_L), \\ p_m = p_R, &\qquad u_m = u_R. \end{aligned} \tag{4.13}$$

The differentiation of (4.13) with respect to ξ_L implies

$$\begin{pmatrix} \sigma_P^2 & (2\sigma_P - p'_R) \\ \{\bar{p}_v^m - \frac{1}{2}p_e^m(p_R + p_L)\} & -p'_R(1 + \frac{1}{2}p_e^m(v_m - v_L)) \end{pmatrix} \begin{pmatrix} \dot{v}_m \\ \dot{\xi}_R \end{pmatrix}$$
$$= \begin{pmatrix} -p'_L + 2\sigma_P + \sigma_P^2 v'_L \\ -p_e^m e'_L + \frac{1}{2}p_e^m p'_L(v_m - v_L) - \frac{1}{2}p_e^m(p_R + p_L)v'_L \end{pmatrix}. \tag{4.14}$$

Denote the determinant of the matrix on the left-hand side as D_2. As in the case of the forward phase boundary, the slope of the entropy rate is determined by $\dot{\Phi}_P$, which is given by

$$\dot{\Phi}_P = \dot{\sigma}_P(s_m - s_L) + \sigma_P(\dot{s}_m - \dot{s}_L)$$
$$= \{(1 - \dot{\xi}_R) - \sigma_P(\dot{v}_m - \dot{v}_L)\}\left(\frac{s_m - s_L}{v_m - v_L}\right) + \sigma_P(\dot{s}_m - \dot{s}_L). \tag{4.15}$$

Since $s_R = s_m \neq s_L$ at $\xi_L = 0$ and

$$\dot{\xi}_R|_{\xi_L = 0} = \frac{p'_L}{p'_R}\bigg|_{\xi_L = 0, \xi_R = 0} = -\frac{\sqrt{-p_v^0}}{\sqrt{-p_v^1}} < 0, \tag{4.16}$$

the term $(1 - \dot{\xi}_R)$, which is positive, determines the slope of $\dot{\Phi}_P$. The relation we have is

$$\begin{aligned} &\dot{\Phi}_P > 0 \quad \text{at} \quad \xi_L = 0 \text{ if } s_1 = s_R = s_m > s_L = s_0 \text{ at } \xi_L = 0, \\ &\dot{\Phi}_P < 0 \quad \text{at} \quad \xi_L = 0 \text{ if } s_m < s_L \text{ at } \xi_L = 0. \end{aligned} \tag{4.17}$$

Using the analogous argument as in the case of the forward phase boundary, we can show, from (4.13b), that ξ_L is negative if we have the backward phase boundary.

Now, we summarize what we have discussed in the case where $s_0 \neq s_1$. Near $\xi_L = 0$, we have

$$\Phi(\xi_L) = \Phi(0) + \dot{\Phi}(0)\xi_L + O(\xi_L^2).$$

Also, $\Phi(0) = 0$, $\dot{\Phi}(0) = \dot{\Phi}_P(0)$. From (4.12) and (4.17) we see that if $s_1 > s_0$, $\dot{\Phi}(0) > 0$, and if $s_1 < s_0$, $\dot{\Phi}(0) < 0$. Therefore, we obtain the following

THEOREM 4.1. *Suppose that there exists a one-parameter family of solutions of the form in Fig.* 2, 3, *or* 4 *for the Riemann problem with* (3.2). *If* $s_1 > s_0$, *the derivative of the entropy rate* $\dot{\Phi}(\xi_L)$ *is positive at* $\xi_L = 0$. *On the other hand, if* $s_1 < s_0$, $\dot{\Phi}(\xi_L)$ *is negative at* $\xi_L = 0$. *In these cases, the solution* *of the form in Fig. 4 (the stationary phase boundary) will not be observed, since it does not minimize Φ.*

The case where $s_1 = s_0$ is delicate. In this case, $\dot{\Phi}_P(\xi_L)$, the derivative of the entropy rate, becomes zero at $\xi_L = 0$. To see if Φ attains a relative minimum at $\xi_L = 0$, we compute the second derivatives of Φ_P with respect to ξ_L for both cases (Figs. 2 and 3) and see if they are positive near $\xi_L = 0$ ($\Phi_B = \Phi_F = O(\xi_L^3)$).

First, we consider the case where we have the forward phase boundary. The second derivative of Φ_P is given by

$$\ddot{\Phi}_P = \ddot{\sigma}_P(s_R - s_m) + 2\dot{\sigma}_P(\dot{s}_R - \dot{s}_m) - \sigma_P(\ddot{s}_R - \ddot{s}_m).$$

Since $s_R = s_m$ and $\sigma_P = 0$ at $\xi_L = 0$ and $\ddot{\sigma}_P$, $\ddot{s}_R$, and $\ddot{s}_m$ are $O(1)$, the sign of $\dot{\sigma}_P(\dot{s}_R - \dot{s}_m)$ will determine the sign of $\ddot{\Phi}_P$. From (4.8), $\dot{\sigma}_P$ is positive at $\xi_L = 0$. From (4.11)

$$p_s^R \dot{s}_R = \dot{p}_R - p_v^R \dot{v}_R = (p'_R - p_v^R v'_R)\,\dot{\xi}_R.$$

The relation (3.10) implies $\dot{s}_R$ is zero at $\xi_L = 0$. To see the sign of $\dot{s}_m$, we substitute $\dot{v}_m$ in (4.6) into (4.11). Then,

$$\begin{aligned}
p_s^m \dot{s}_m|_{\xi_L=0} &= \frac{1}{D_1}\,[-p'_L p'_R\{\bar{p}_v^m - \tfrac{1}{2}p_e^m(p_R + p_L)\} \\
&\quad - p_v^m p'_L\{p_e^m e'_R + p_e^m(v_R - v_m)\,p'_R + \tfrac{1}{2}p_e^m(p_R + p_L)v'_R - p'_R\} \\
&\quad + O(\sigma_P)]|_{\xi_L=0} \\
&= \frac{1}{D_1}\,\{-p_v^0 p_e^0 p'_0 p'_1(v_1 - v_0)\}.
\end{aligned} \tag{4.18}$$

To obtain this, we used the relations

$$\begin{gathered}
p_R = p_L \qquad \text{at} \quad \xi_L = 0, \\
p_v = -pp_e + \bar{p}_v, \\
e'_R|_{\xi_L=0} = \frac{p_1}{\sqrt{-p_v^1}}, \qquad v'_R|_{\xi_L=0} = -\frac{1}{\sqrt{-p_v^1}}.
\end{gathered} \tag{4.19}$$

Since D_1, p_s^m, p_e^0, p'_1, and $(v_1 - v_0)$ are positive and p_v^0 and p'_0 are negative, $\dot{s}_m$ is negative at $\xi_L = 0$. Therefore, $\ddot{\Phi}$ has positive sign at $\xi_L = 0$.

Next, we discuss the case where we have the backward phase boundary. In this case, the second derivative of Φ_P is given by

$$\ddot{\Phi}_P = \ddot{\sigma}_P(s_R - s_m) + 2\dot{\sigma}_P(\dot{s}_R - \dot{s}_m) + \sigma_P(\ddot{s}_R - \ddot{s}_m).$$

As before, the sign of $\dot{\sigma}_{\mathrm{P}}(\dot{s}_{\mathrm{R}}-\dot{s}_m)$ determines the sign of $\ddot{\Phi}_{\mathrm{P}}$ at $\xi_{\mathrm{L}}=0$. From (4.15), $\dot{\sigma}_{\mathrm{R}}$ is positive at $\xi_{\mathrm{L}}=0$. Using (3.10) and the relation

$$\dot{p}_{\mathrm{L}}=p'_{\mathrm{L}}=p_v^{\mathrm{L}}\dot{v}_{\mathrm{L}}+p_s^{\mathrm{L}}\dot{s}_{\mathrm{L}}=p_v^{\mathrm{L}}v'_{\mathrm{L}}+p_s^{\mathrm{L}}s'_{\mathrm{L}},$$

we see $\dot{s}_{\mathrm{L}}=0$ at $\xi_{\mathrm{L}}=0$. To find out the sign of $\dot{s}_m$, we use (4.14) and the relation

$$\dot{p}_m=\dot{p}_{\mathrm{R}}=p'_{\mathrm{R}}\dot{\xi}_{\mathrm{R}}=p_v^m\dot{v}_m+p_s^m\dot{s}_m.$$

Then,

$$\begin{aligned}p_s^m\dot{s}_m|_{\xi_{\mathrm{L}}=0}&=\frac{1}{D_2}\,[p'_{\mathrm{R}}\,p'_{\mathrm{L}}\{\bar{p}_v^m-\tfrac{1}{2}p_e^m(p_{\mathrm{R}}+p_{\mathrm{L}})\}-p_v^m\,p'_{\mathrm{R}}\,p'_{\mathrm{L}}\\&\quad-p_v^m\,p_e^m\,p'_{\mathrm{R}}\,p'_{\mathrm{L}}(v_m-v_{\mathrm{L}})+p'_{\mathrm{R}}\,p_e^m\{e'_{\mathrm{L}}+\tfrac{1}{2}(p_{\mathrm{R}}+p_{\mathrm{L}})v'_{\mathrm{L}}\}]|_{\xi_{\mathrm{L}}=0}\\&=-\frac{1}{D_2}\,p_v^1\,p_e^1\,p'_1\,{}'_0(v_1-v_0).\end{aligned}\tag{4.20}$$

The similar relations to (4.19) have been used to derive the above relation. As p_s^m, p_e^1, p'_{R}, and (v_1-v_0) are positive and D_2, p_v^1, and p'_0 are negative, $\dot{s}_m$ is positive at $\xi_{\mathrm{L}}=0$. Therefore, $\ddot{\Phi}$ is positive at $\xi_{\mathrm{L}}=0$.

Now, we summarize what we have discussed in the case where $s_0=s_1$. Near $\xi_{\mathrm{L}}=0$, we have

$$\Phi(\xi_{\mathrm{L}})=\Phi(0)+\dot{\Phi}(0)\xi_{\mathrm{L}}+\tfrac{1}{2}\ddot{\Phi}(0)\,\xi_{\mathrm{L}}^2+O(\xi_{\mathrm{L}}^3).$$

Also, $\Phi(0)=0$, $\dot{\Phi}(0)=\dot{\Phi}_{\mathrm{P}}(0)=0$. As $\ddot{\Phi}(0)$ is positive from the above argument, we have the following

THEOREM 4.2. *Suppose that there exists a one-parameter family of solutions of the form in Fig.* 2, 3, *or* 4 *for the Riemann problem with* (3.2). *If* $s_0=s_1$, *the entropy rate* $\Phi(\xi_{\mathrm{L}})$ *attains a relative minimum at* $\xi_{\mathrm{L}}=0$. *In this case the stationary phase boundary* (*Fig.* 4) *will be observed.*

5. AN EXAMPLE OF NONTRIVIAL SOLUTION

Theorems 4.1 and 4.2 agree with the results in the classical thermodynamics. Namely, in an isolated system the entropy of the final state can never be less than of the initial state. Therefore, to enforce the applicability of the entropy rate admissibility criterion, it is desirable to show that, in the case where $s_0\neq s_1$, there exists a nontrivial solution which minimizes the entropy rate among the solutions assumed in Figs. 2 and 3 [12]. For this purpose, we impose additional assumptions:

$(V-1)$ along shock curves the entropy s increases,

$(V-2)$ $p_{vv}(v, s)$ is positive in the regions containing (v_0, s_0) and (v_1, s_1).

The first assumption is physically accepted and also a consequence of Liu's extended entropy condition [7] (or Lax's entropy condition [11]). The second assumption is equivalent to the condition of genuine nonlinearity [7, 11].

First, we discuss how the sign of $\dot{\Phi}_{\mathrm{P}}$ in (4.8) and (4.15) changes as a function of ξ_{L}.

LEMMA 5.1. *Assume that* $(V-1)$ *holds. Then, if* $s_1 < s_0$ *and they are close,* $\dot{\Phi}_{\mathrm{P}}$ *in* (4.8) *changes the sign from negative to positive as* ξ_{L} *increases from zero. If* $s_1 > s_0$ *and they are close,* $\dot{\Phi}_{\mathrm{P}}$ *in* (4.15) *changes sign from positive to negative as* ξ_{L} *decreases from zero.*

Proof. Consider the case where $s_1 < s_0$. In this case we note that ξ_{L} increases from zero and $\dot{\xi}_{\mathrm{R}}$ is negative at $\xi_{\mathrm{L}} = 0$ (from (4.10)). Therefore, there is an interval of ξ_{L} in which ξ_{R} decreases as ξ_{L} increases. This interval depends on the value of s_1. Denote by $[0, \xi_1]$ the smallest interval of ξ_{L} on which ξ_{R} decreases for $s_1 \in [s_0 - \varepsilon, s_0]$, where ε is a small positive constant.

Next, we see how Φ_{P} in (4.8) changes sign as ξ_{L} increases from zero. Notice that, near $\xi_{\mathrm{L}} = 0$, $(s_{\mathrm{R}} - s_m)$ in (4.8) can be written in the following way:

$$\begin{aligned} s_{\mathrm{R}}(\xi_{\mathrm{L}}) - s_m(\xi_{\mathrm{L}}) &= s_1 - s_0 + (\dot{s}_{\mathrm{R}}(0) - \dot{s}_m(0))\xi_{\mathrm{L}} + O(\xi_{\mathrm{L}}^2), \\ \dot{s}_{\mathrm{R}}(\xi_{\mathrm{L}}) - \dot{s}_m(\xi_{\mathrm{L}}) &= (\dot{s}_{\mathrm{R}}(0) - \dot{s}_m(0)) + O(\xi_{\mathrm{L}}). \end{aligned} \tag{5.1}$$

Let us examine the signs of $\dot{s}_{\mathrm{R}}(0)$ and $\dot{s}_m(0)$, starting from $\dot{s}_{\mathrm{R}}(0)$. If the forward wave is a rarefaction wave, it is obvious that $\dot{s}_{\mathrm{R}}(0) = 0$. If the forward wave is a shock, as $s_{\mathrm{R}} = O(\xi_{\mathrm{R}}^3) = O(\xi_{\mathrm{L}}^3)$, $\dot{s}_{\mathrm{R}}(0) = 0$. Now, we examine the sign of $\dot{s}_m(0)$. By (4.18), $\dot{s}_m(0)$ is given by

$$\dot{s}_m(0) = \frac{1}{D_1}\{-p_v^m p_e^m p'_{\mathrm{L}} p'_{\mathrm{R}}(v_{\mathrm{R}} - v_m)\}|_{\xi_{\mathrm{L}}=0} = -p_e^0 p'_0(v_1 - v_0).$$

As has been discussed in Remark 4.2, we treat the case where v_1 is greater than v_0. In this case, $\dot{s}_m(0) < 0$. Therefore, if s_1 is sufficiently close to s_0, there exists the value of $\xi_{\mathrm{L}}(>0)$ such that $s_{\mathrm{R}} = s_m$, $\dot{s}_{\mathrm{R}} > \dot{s}_m$, and $\xi_{\mathrm{L}} \in (0, \xi_1]$. Denote by $\bar{\xi}_{\mathrm{L}}$ the above value of ξ_{L}. Since at $\xi_{\mathrm{L}} = \bar{\xi}_{\mathrm{L}}$

$$u_m - u_{\mathrm{R}} = (u_{\mathrm{L}} - u_0) - (u_{\mathrm{R}} - u_1) = \xi_{\mathrm{L}} - \xi_{\mathrm{R}} > 0,$$

(4.4b) implies that σ_P is positive. Therefore, from (4.8), $\dot{\Phi}_P$ is positive at $\xi_L = \bar{\xi}_L$. On the other hand, from (4.12b), $\dot{\Phi}_P$ is negative at $\xi_L = 0$. Hence, if $s_1 < s_0$ and they are close, $\dot{\Phi}_P$ changes sign from negative to positive as ξ_L increases. The case where $s_1 > s_0$ is proved in a similar manner. Q.E.D.

Next, we discuss a consequence of assumption $(V-2)$. If we differentiate the Rankine–Hugoniot condition for the backward shock with respect to ξ_L $(= u - u_0)$, we see

$$\dot{\sigma}(u-u_0) + \sigma = p_v \dot{v} + p_s \dot{s},$$
$$\dot{\sigma}(v-v_0) + \sigma \dot{v} = -1.$$

Eliminating $\dot{\sigma}$ and using (3.10), we obtain

$$p_s \dot{s} = \frac{p_e(u-u_0)(\sigma^2 + p_v)}{p_v - \sigma^2 + p_e(p-p_0)}.$$

Set $B = \sigma^2 + p_v$. Then, using $s - s_0 = O(\xi_L^3)$ and l'Hôpital's rule, we find

$$\lim_{\xi_L \to 0} B = 0, \qquad \lim_{\xi_L \to 0} \dot{B} = \tfrac{1}{2} p_{vv} \dot{v}.$$

As $\dot{v} = -1/\sqrt{-p_v}$ at $\xi_L = 0$, the entropy decreases as a function of ξ_L along the forward shock curve at $\xi_L = 0$. Therefore, by means of $(V-1)$ and $(V-2)$, state (u_L, v_L, e_L) is joined to (u_0, v_0, e_0), on the right, by a backward shock if $u_L < u_0$ and by a backward rarefaction wave if $u_L > u_0$. Similarly, state (u_R, v_R, e_R) is joined to (u_1, v_1, e_1) by a forward shock if $u_R > u_1$ and by a forward rarefaction wave if $u_R < u_1$. See [7] for the details.

Now, as an example, consider case (i) in Section 4, namely, $u_0 = u_1$, $p_0 = p_1$, and $s_0 > s_1$. In this case we have the following

THEOREM 5.1. *Assume that $(V-1)$ and $(V-2)$ hold. Then, if $u_0 = u_1$, $p_0 = p_1$, and $s_0 > s_1$ and they are close, there is a nontrivial solution of the form in Fig. 2 for which the entropy rate Φ is locally minimized.*

Proof. As $s_0 > s_1$, from (4.12) and (4.17) we see that $\dot{\Phi}_P$ is negative at $\xi_L = 0$. As $\dot{\Phi}_F$ and $\dot{\Phi}_B$ are zero at $\xi_L = 0$, the entropy rate Φ has smaller value when $\xi_L > 0$. In this case, as was observed, (u_0, v_0, e_0) is joined to (u_L, v_L, e_L) by the backward rarefaction wave, and (u_R, v_R, e_R) is joined to (u_1, v_1, e_1) by the forward rarefaction wave. Therefore, $\Phi = \Phi_P$ and Lemma 4.1 implies that there is a local minimum of Φ when $\xi_L > 0$. In this case the phase boundary moves forward. Q.E.D.

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REFERENCES

1. C. M. DAFERMOS, The entropy rate admissibility criterion for solutions of hyperbolic conservation laws, *J. Differential Eequations* **14** (1973), 202–212.
2. C. M. DEFERMOS, The entropy rate admissibility criterion in thermoelasticity, *Atti Accad. Naz. Lincei Rend. cl. Sci. Fis. Mat. Natur. (8)*, (1974), 113–119.
3. H. HATTORI, The Riemann problem for a van der Waals fluid with entropy rate admissibility criterion—Isothermal case, *Arch. Rational Mech. Anal.*, in press.
4. M. SLEMROD, Admissibility criteria for propagating phase boundaries in a van der Waals fluid, *Arch. Rational Mech. Anal.* **81** (1983), 301–315.
5. M. SLEMROD, Dynamic phase transitions in a van der Waals fluid, *J. Differential Equations*, in press.
6. J. SERRIN, Phase transitions and interfacial layers for van der Waals fluid, *in* "Proceedings, SAFA IV Conference, Recent Methods in Nonlinear Analysis and Applications" (A. Canfora, S. Rionero, C. Sbordone, and C. Trombetti, Eds.), Naples, 1980.
7. T. P. LIU, The Riemann problem for general systems of conservation laws, *J. Differential Equations* **18** (1975), 218–234.
8. L. HSIAO, The entropy rate admissibility criterion in gas dynamics, *J. Differential Equations* **38** (1980), 226–238.
9. H. HATTORI, On the properties of entropy rate in the entropy rate admissibility criterion, *Quart. Appl. Math.*, in press.
10. E. FERMI, "Thermodynamics," Dover, New York, 1956.
11. P. D. LAX, "Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves," CBMS Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1973.
12. M. SLEMROD, personal communication.